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On graphs which contain each tree of given size

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Abstract

Dobson (1994) conjectured that if G is a graph with girth no less than $2t+1$ and minimum degree no less than k/t and $\Delta(T)$, then G contains each tree T of size k.

It is known that this conjecture holds for $t=1$ and $t=2$.

We prove it in the case $t=3$.

1. Terminology

We shall use standard graph theory notation. We consider only finite, undirected graphs of order $n=|V(G)|$ and size $e(G)=|E(G)|$. All graphs will be assumed to have neither loops nor multiple edges. The minimum and the maximum degrees of a vertex in the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We shall need some additional definitions in order to formulate the results.

Recall first that S_n denote the star on n vertices.

In a graph G, a vertex of degree one will be called an *end-vertex* or *a pendent vertex*. A pendent vertex in a tree is also called *a leaf.* If a vertex has a neighbor which is an end-vertex, it is referred as its *end-neighbor*.

An edge incident with an end-vertex is an *end-edge* or a *pendent edge. A penultimate vertex* in a graph G is a vertex having at least one end-neighbor and exactly one neighbor which is not end-vertex. In other words, a penultimate vertex is an end-vertex in the graph obtained from G by removing all its end-vertices. Note that any nonstar tree has at least two penultimate vertices.

A caterpillar is a tree such that if we remove all its pendent vertices, the resulting graph is a path. The caterpillars are the trees having exactly two penultimate vertices.

We shall need the following simple lemma.

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Lemma 1. *If a connected graph of order s has s' vertices which have pairwise distance at least 5, then $s' \leqslant s/3$.*

This lemma can be considered as a very special case of the following result of Brass [2].

Theorem 2. *If G is an r-connected graph and there are k vertices which have pairwise distance at least d, then G has at least*

$$k\left(r\left\lfloor\frac{d-1}{2}\right\rfloor+1\right)+\left(\frac{1+(-1)^d}{2}\right)r$$

vertices, and this bound is sharp.

2. Results

The following conjecture is due to Dobson [3].

Conjecture 3. Let G be a graph with girth $g \geqslant 2t+1$ and let T be a tree of size k. If $\delta(G) \geqslant k/t$ and $\delta(G) \geqslant \Delta(T)$, then G contains T.

The fact that the conjecture holds for $t=1$ is well known. Usually, it is presented in the following form.

Theorem 4. *Let T be a tree of size k. If $\delta(G) \geqslant k$, then G contains T.*

For $t=2$, Brandt and Dobson proved actually [1] the following somewhat stronger result.

Theorem 5. *Let G be a graph with girth at least 5 and let T be a tree of size k. If $\delta(G) \geqslant k/2$ and $\Delta(G) \geqslant \Delta(T)$, then G contains T.*

As a direct consequence of the above theorem Brandt and Dobson proved in [1] that:

Theorem 6. *Let G be a graph of order n and with girth at least 5. If the size $e(G) > \frac{1}{2}n(k-1)$, then G contains each tree of size k.*

Remark. An earlier, considerably longer proof by Dobson can be found in [3].

The above theorem is a special case of another, well-known (and still open) conjecture concerning the graphs which contain each tree of given size, namely the following conjecture made by Erdős and Sós in 1963.

Conjecture 7. If G is a graph of order n and of size $e(G) > \frac{1}{2}n(k-1)$, then G contains every tree T of size k.

It should be mentioned here that in the case of the Erdös–Sós conjecture the following result improving Theorem 6 holds [5] (see also [6] for some other partial results on this conjecture).

Theorem 8. *Suppose that a graph G of order n does not contain the cycle C_4. If the size $e(G) > \frac{1}{2}n(k-1)$, then G contains each tree of size k.*

The main result of this paper is the case $t = 3$ of Conjecture 3. We formulate it below as a theorem. The proof is given in the next section.

Theorem 9. *Let G be a graph with girth at least 7 and let T be a tree of size k. If $\delta(G) \geqslant k/3$ and $\delta(G) \geqslant \Delta(T)$, then G contains T.*

Remark. It is worth mentioning that Dobson [4] has proven, for $t \geqslant 4$, that for each $\varepsilon > 0$, there exists n such that every graph G of order no less than n, with girth no less than $2t+1$ and $\delta(G) \geqslant k/t$, contains every tree T of size k with $(1-\varepsilon)k/t \geqslant \Delta(T)$.

3. Proof of Theorem 9

The proof is by induction on k. It is easy to see that the theorem is true for small values of k.

Let k be the smallest integer such that there are two graphs G and T satisfying $\delta(G) \geqslant k/3$, $\delta(G) \geqslant \Delta(T)$, $g(G) \geqslant 7$, T is a tree with k edges, and G does not contain T.

Let u be a penultimate vertex of T with minimum number of end-neighbors r. By removing these leaves we get the tree T' with u as a leaf. By the choice of T, we can consider the graph T' as a subgraph of G.

Denote by v the (unique) neighbor of u in T' and let $T'' = T' \setminus \{u\}$. For a vertex $x \in V(T'')$ we shall denote by $B(x)$ the set consisting of the vertex x and all its descendants (i.e. its sons, the sons of its sons and so on) with respect to the tree T'' considered as a tree rooted at v.

We shall consider two main cases.

Case 1: T has at least three penultimate vertices.

Let w_1, w_2 be two other penultimate vertices of T and let W_1, W_2 be the sets of their end-neighbors, respectively. We put $|W_1| = r_1$ and $|W_2| = r_2$. By the choice of u we have $r \leqslant r_1$ and $r \leqslant r_2$.

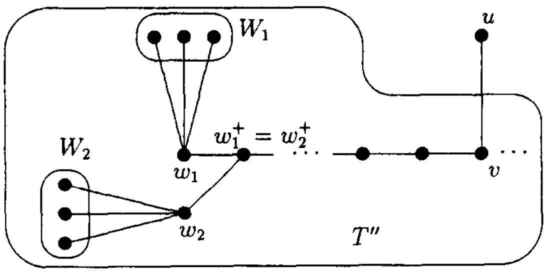

Fig. 1. Case 1(a).

Without loss of generality, we may choose the vertex w_1 with $\mathrm{dist}_{T''}(v, w_1)$ as great as possible. Let P be the path of T'' joining w_1 with v.

For a vertex $y \in V(T'')$, $y \neq v$ denote by y^+ the neighbor of y lying on the path joining y with v in T''.

Suppose first that

Case 1(a): There is a penultimate vertex w_2 of T'', $w_1 \neq w_2$, such that $w_1^+ = w_2^+$ (cf. Fig. 1).

By the choice of w_1, it is easy to see that the set $B = B(w_1^+) = B(w_2^+)$ can receive at most one edge from u without creating a cycle of length less than seven. We have $|B| \geqslant 2r + 3$.

Observe also that, by definition of w_1 and w_2, the graph obtained from T'' by removing B remains connected. Denote it by T''' and let s be the number of its vertices. We have $s \leqslant k + 1 - r - 1 - 2r - 3 = k - 3r - 3$.

Suppose that there are s_1 vertices of T''' adjacent to u in G. By Lemma 1 we have: $s_1 \leqslant s/3$.

Finally, there are at most $1 + (k - 3r - 3)/3$ edges between u and $V(T'')$.

On the other hand, it is easy to see that u has at most $r - 1$ neighbors in G outside of $V(T'')$. For, otherwise, T'' together with u and its neighbors would be a tree subgraph of G isomorphic to T.

Thus, by assumptions, at least $k/3 - r + 1$ neighbors of u are in $V(T'')$.

Therefore, the following inequality must hold:

$$\frac{k}{3} - r + 1 \leqslant 1 + \frac{k - 3r - 3}{3},$$

which is impossible.

Case 1(b): There is a penultimate vertex w_2 of T'', $w_1 \neq w_2$, such that w_2^+ is not on P.

We choose the third penultimate vertex w_2 with the distance from the path P as great as possible.

Observe that, by definition of w_1 and w_2, the set $B(w_i^+)$, $i = 1,2$ can receive in G at most one edge from u without creating a cycle of length less than seven. We have $|B(w_i^+)| \geqslant r+2$, $i = 1,2$.

Moreover, the graph obtained from T'' by removing $B(w_1^+)$ and $B(w_2^+)$ remains connected.

As above, denote it by T''' and let s be the number of its vertices. We have: $s \leqslant k+1-r-1-2r-4 = k-3r-4$.

Therefore, the total number of possible edges between u and $V(T'')$ in G is now no greater than $1+1+s/3$. Finally we have

$$\frac{k}{3} - r + 1 \leqslant 2 + \frac{k-3r-4}{3}.$$

Once again we get a contradiction.

Case 1(c): For each penultimate vertex w_2 of T'', $w_1 \neq w_2$, we have $w_2^+ \in V(P)$.

We may, of course, assume that the situation of the Case 1(a) does not hold. Suppose first that there is a penultimate vertex w_2 of T'', $w_1 \neq w_2$, such that $w_2^+ = w_1^{++}$. Observe that in this case the set $B(w_1^{++})$ contains at least $2r+4$ vertices and there are at most two possible edges between it and u. So, we get exactly the same inequality as in Case 1(b). Thus, we may assume that the neighbors of the vertex w_1^{++} in T are either its neighbors on P or its leaves. This implies that there is at most one edge possible between u and $B(w_1^{++})$. Note also that $|B(w_1^{++})| \geqslant r+3$ (cf. Fig. 2).

It is easy to see that the set $B(w_2)$ can receive at most one edge from u without creating a cycle of length less than seven. We have $|B(w_2)| \geqslant r+1$.

As above, we consider the graph T''' of order s obtained from T'' by removing $B(w_2)$ and $B(w_1^{++})$.

Therefore, the total number of possible edges between u and $V(T'')$ is now no greater than $1+1+s/3$, where $s \leqslant k+1-r-1-r-3-r-1$. Finally, we have again

$$\frac{k}{3} - r + 1 \leqslant 2 + \frac{k-3r-4}{3},$$

a contradiction.

Case 2. T has only two penultimate vertices.

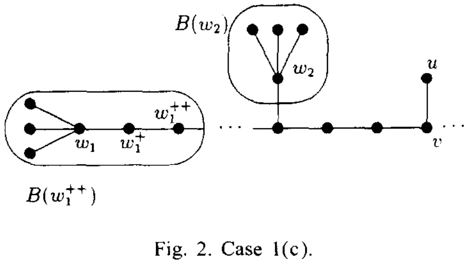

Fig. 2. Case 1(c).

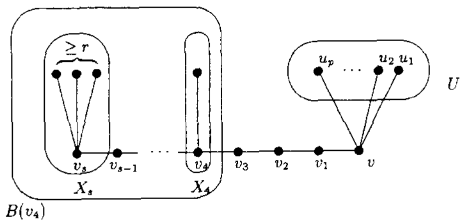

Fig. 3. Case 2.

As already noticed, T is a caterpillar. As before, denote by P the path of T'' joining v with the second penultimate vertex of T. We put $P = vv_1v_2v_3 \cdots v_s$.

Denote by U the set of those neighbors of v in G which are either its leaves in T' or are outside of $V(T'')$. Let $|U| = p$. Since, $u \in U$ we have $p \geqslant 1$. We put $d_G(v) = p + q + 1$. This means that there are q vertices of T'' other than v_1 adjacent to v in G. It is easy to see that no vertex of U has r (or more) neighbors outside of $V(T'')$. Otherwise, T'' together with this vertex and its neighbors would be a tree subgraph of G isomorphic to T.

Denote by X_i, $i = 1, 2, \ldots, s$, the set containing the vertex v_i together with its end-neighbors in T (cf. Fig. 3).

It is easy to see that there is no edge between U and $X_1 \cup X_2 \cup X_3$ in G. Moreover, an edge between U and X_4 is possible only if X_4 contains a leaf.

We shall estimate s' the number of possible edges between U and $B(v_4)$, that is the part of the caterpillar T'' which contains the vertices $v_4v_5 \cdots v_s$ together with its leaves.

Each set X_i, $i \geqslant 4$, can receive at most one edge from U. For, otherwise, we would have a cycle of length six or less. So, we get our first estimation

(i) $s' \leqslant k - 2r - 4$.

Observe that if $q \geqslant 1$, we can obtain a better estimation. It is easy to see that if there is an edge between X_i and v, then there is no edge between $X_{i-1} \cup X_i \cup X_{i+1}$ and U. Hence, each edge between $B(v_4)$ and v excludes three sets X_i, except maybe, for the case where $i = s$. Moreover, for two such edges the excluded triples do not overlap. We have thus at least $3(q-1) + 2 = 3q - 1$ sets X_i without edge going to U. So, we get our second estimation

(ii) $s' \leqslant k - 2r - 3 - 3q$.

As in the previous case, it is easy to see that each vertex u_j of U sends at most $r - 1$ edges outside of T'' and one edge to v. Thus, at least $d_G(u_j) - r$ edges are sent to $B(v_4)$. So, there are at least $p(\delta - r)$ edges between U and $B(v_4)$, where $\delta = \delta(G)$. We get

(iii) $p(\delta - r) \leqslant s'$.

Hence, using (ii) and the fact that $p + q + 1 = d_G(v) \geqslant k/3$, we have

(iv) $p(\delta - r - 3) \leqslant -2r$.

Therefore, if $\delta - r - 3 \geqslant 0$ we get a contradiction. On the other hand, since $\delta \geqslant \Delta(T)$ and $d_T(u) = r + 1$, we have $\delta \geqslant r + 1$. Thus, two cases remain to be considered.

Case 2(a): $\delta = r + 1$.

Since $\delta \geqslant k/3$, we have $k \leqslant 3r+3$. From (iv) we get $p \geqslant r$. Using these inequalities and (i) instead of (ii) in (iii) we obtain

$$r \leqslant p \leqslant k - 2r - 4 \leqslant 3r + 3 - 2r - 4 \leqslant r - 1,$$

which is a contradiction.

Case 2(b): $\delta = r + 2$.

The argument in this case is similar as above. Since $\delta \geqslant k/3$, we have $k \leqslant 3r + 6$. From (iv) we get $p \geqslant 2r$. Using these inequalities and (i) instead of (ii) in (iii) we obtain

$$4r \leqslant 2p \leqslant k - 2r - 4 \leqslant 3r + 6 - 2r - 4 \leqslant r + 2,$$

which is a contradiction since $r \geqslant 1$.

This finishes the proof of our theorem. □

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